1. First-Order Ordinary Differential Equations

1.1 Fundamental Definitions and Classifications

- **Ordinary Differential Equation (ODE):** An equation involving an unknown function of a single independent variable and one or more of its derivatives.
- **Order:** Determined by the highest-order derivative appearing in the equation.
- **Degree:** The power (exponent) of the highest-order derivative appearing in the equation, after the equation has been written in polynomial form with respect to its derivatives.
- **Explicit and Implicit Forms:** An explicit first-order ODE takes the form $y' = f(x, y)$, whereas the implicit form is represented as $F(x, y, y') = 0$.
- **General Solution:** A family of solutions containing one arbitrary constant, traditionally denoted by C.
- **Particular Solution:** A solution obtained by assigning a specific, unique value to the arbitrary constant, typically derived from initial conditions.
- **Initial Value Problem (IVP):** A first-order differential equation paired with an initial condition: $y' = f(x, y)$, $y(x_0) = y_0$.

1.2 Physical Models and Geometric Concepts

- **Population Growth Model:** $dP/dt = kP$ (growth constant $k > 0$), with the general solution: $P(t) = P_0 * e^{(kt)}$
- **Radioactive Decay Model:** $dN/dt = -\lambda N$ (decay constant $\lambda > 0$), with the general solution: $N(t) = N_0 * e^{(-\lambda t)}$
- **Newton's Law of Cooling:** $dT/dt = -k(T - T_s)$ (surrounding temperature T_s, cooling constant $k > 0$), with the general solution: $T(t) = T_s + (T_0 - T_s)e^{(-kt)}$
- **Isocline:** For $y' = f(x, y)$, an isocline is a curve along which every solution has the same constant slope c, defined implicitly by: $f(x, y) = c$

1.3 Picard-Lindelöf Existence and Uniqueness Theorem

Theorem Formulation: For an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ defined on a rectangle $R = \{ (x, y) : |x - x_0| \leq a, |y - y_0| \leq b \}$:

- **Existence:** If $f(x, y)$ is continuous on R, then a solution $y(x)$ exists.
- **Uniqueness:** If both $f(x, y)$ and $\partial f/\partial y$ are continuous and bounded on R, then the solution $y(x)$ is unique. This continuity of $\partial f/\partial y$ guarantees the Lipschitz condition: $|f(x, y_2) - f(x, y_1)| \leq M |y_2 - y_1|$.

1.4 Exact Differential Equations

Exactness Definition: A differential form $M(x, y)dx + N(x, y)dy = 0$ is exact in a simply connected region R if there exists a differentiable function $F(x, y)$ such that $dF = Mdx + Ndy$.

> **Exactness Form:** $M(x, y)\, dx + N(x, y)\, dy = 0$
> **Exactness Condition:** $\partial M/\partial y = \partial N/\partial x$
> **Implicit General Solution:** $F(x, y) = C$

1.5 Integrating Factors and Reduction to Exactness

Integrating Factor Theorem: If the differential form is not exact, multiplying by an integrating factor μ(x, y) yields an exact equation: ∂(μ M)/∂y = ∂(μ N)/∂x.

- **Case 1:** Integrating factor depending only on x. If (M_y - N_x) / N = f(x) is a function of x alone:

> **Formula:** μ(x) = exp(∫ f(x) dx) = exp(∫ [(∂M/∂y - ∂N/∂x) / N] dx)

- **Case 2:** Integrating factor depending only on y. If (N_x - M_y) / M = g(y) is a function of y alone:

> **Formula:** μ(y) = exp(∫ g(y) dy) = exp(-∫ [(∂M/∂y - ∂N/∂x) / M] dy)

- **Case 3:** Integrating factor depending on the product z = xy. If (M_y - N_x) / (yN - xM) = h(z) is a function of z alone:

> **Formula:** μ(z) = exp(∫ h(z) dz)

- **Case 4:** Integrating factor depending on the sum z = x + y. If (M_y - N_x) / (N - M) = f(z) is a function of z alone:

> **Formula:** μ(z) = exp(∫ f(z) dz)

- **Case 5:** Homogeneous Equations. If M dx + N dy = 0 is homogeneous and Mx + Ny ≠ 0, then:

> **Formula:** μ(x, y) = 1 / (M*x + N*y)

- **Case 6:** Equations of the form f_1(xy)y dx + f_2(xy)x dy = 0. If Mx - Ny ≠ 0, then:

> **Formula:** μ(x, y) = 1 / (M*x - N*y)

1.6 Integrating Factors of the Form $x^a y^k$

Method of $x^a y^k$: For non-exact equations of the form: $x^a y^b$ (my dx + nx dy) + $x^c y^d$ (py dx + qx dy) = 0 with np ≠ mq.

- Multiply the entire ODE by $x^a y^k$ and equate partial derivatives to find h and k:

> **System Eq 1:** n*h - m*k = m(b + 1) - n(a + 1)
> **System Eq 2:** q*h - p*k = p(d + 1) - q(c + 1)

1.7 Linear Differential Equations (Lagrange Form)

Standard Form: A first-order differential equation is linear in y if it is of the form:

> **Linear in y:** dy/dx + P(x) y = Q(x)

> **Integrating Factor:** $\mu(x) = \exp(\int P(x)\,dx)$
> **General Solution:** $y = e^{\wedge}(-\int P(x)dx)\,[\,\int Q(x)\,e^{\wedge}(\int P(x)dx)\,dx + C\,]$

- **Linear in x:** If treated with x as the dependent variable and y as independent variable:

> **Linear in x:** $dx/dy + P(y)\,x = Q(y)$
> **Integrating Factor:** $\mu(y) = \exp(\int P(y)\,dy)$
> **General Solution:** $x = e^{\wedge}(-\int P(y)dy)\,[\,\int Q(y)\,e^{\wedge}(\int P(y)dy)\,dy + C\,]$

1.8 Bernoulli Differential Equation

Standard Form: A first-order non-linear differential equation is of Bernoulli type if it is of the form:

> **Bernoulli Equation:** $dy/dx + P(x)\,y = Q(x)\,y^{n}$ (where $n \neq 0, 1$)
> **Substitution Variable:** $v = y^{1-n}$
> **Linearized Form:** $dv/dx + (1 - n)\,P(x)\,v = (1 - n)\,Q(x)$